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Distance antimagic labeling of join and corona of two graphs

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Abstract

Let G be a graph of order n. Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. The weight $w_f(v)$ of a vertex v with respect to f is defined by $w_f(v) = \sum_{x \in N(v)} f(x)$, where $N(v)$ is the open neighborhood of v. The labeling f is said to be distance antimagic if $w_f(u) \neq w_f(v)$ for every pair of distinct vertices $u, v \in V(G)$. If the graph G admits such a labeling, then G is said to be a distance antimagic graph. In this paper we investigate the existence of distance antimagic labelings of $G + H$ and $G \circ H$.

Keywords: Distance antimagic labeling; Arbitrarily distance antimagic labeling; Distance magic labeling

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops, multiple edges or isolated vertices. For graph theoretic terminology we refer to West [1].

Most of the graph labeling methods trace their origin to the concept of β-valuation introduced by Rosa [2]. For a general overview of graph labeling we refer to the dynamic survey by Gallian [3].

Let G be a graph of order n. Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. The weight $w_f(v)$ of a vertex v is defined by $w_f(v) = \sum_{x \in N(v)} f(x)$. The labeling f is said to be *distance antimagic* if $w_f(u) \neq w_f(v)$ for every pair of vertices $u, v \in V(G)$. If the graph G admits such a labeling, then G is said to be a *distance antimagic graph*. Many classes of graphs are known to be distance antimagic. For details one may refer to Gallian [3] and Kamatchi and Arumugam [4].

Let $G = (V, E)$ be a graph of order n. Let $f : V \rightarrow \{1, 2, \ldots, n\}$ be a bijection. For $v \in V$, the weight of v is defined to be $w_f(v) = \sum_{x \in N(v)} f(x)$. If $w_f(v) = k$ (a constant) for every $v \in V$, then f is said to be a *distance magic labeling* of the graph G. A graph which admits a distance magic labeling is called a *distance magic graph*. The constant k is called the *distance magic constant*. For more details one may refer to Arumugam et al. [5]

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Definition 1.1. Let G and H be two graphs. The corona $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H and joining each vertex of the ith copy of H to the ith vertex of G.

In [4] Kamatchi and Arumugam proved that the graphs P_n $(n \neq 3)$ and C_n $(n \neq 4)$ are distance antimagic. They also proved that for any graph G the corona $H = G \circ K_1$ is a distance antimagic graph. They posed the following problem:

Problem 1.2. If G is a distance antimagic graph, are $G + K_1$ and $G + K_2$ distance antimagic?

In this paper we solve the above problem in the affirmative. We also prove certain sufficient conditions for the existence of distance antimagic labeling for the join and corona of two graphs.

Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. Then for any positive integer k, we define a bijection $f_k : V(G) \longrightarrow \{k + 1, k + 2, \ldots, k + n\}$ by $f_k(x) = f(x) + k$. It should be noted that if f is a distance antimagic labeling then f_k need not induce distinct vertex weights.

For example, for the path $P_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$, the labeling $f : V \longrightarrow \{1, 2, \ldots, 6\}$ defined by $f(v_i) = i$ is a distance antimagic labeling. However $w_{f_1}(v_6) = w_{f_1}(v_2) = 6$.

These observations lead to the following concept of arbitrarily distance antimagic labeling, which serves as a nice tool in proving that under certain conditions the join of two graphs is distance antimagic.

Definition 1.3. A graph G of order n is said to be arbitrarily distance antimagic if there exists a bijection $f : V \longrightarrow \{1, 2, \ldots, n\}$ such that $w_{f_k}(u) \neq w_{f_k}(v)$ for any two distinct vertices u and v and for any $k \geq 0$. The labeling f with this property is called an arbitrarily distance antimagic labeling of G.

In this paper we use the concept of arbitrarily distance antimagic labeling to prove the existence of distance antimagic labeling of $G + H$ and $G \circ H$ for some graphs G and H.

2. Arbitrarily distance antimagic labeling of graphs

In this section, we discuss the existence of arbitrarily distance antimagic labelings of some graphs.

Proposition 2.1. *Any r-regular distance antimagic graph G is arbitrarily distance antimagic.*

Proof. Let f be a distance antimagic labeling of G. Since $w_{f_k}(u) = w_f(u) + rk$ and $w_f(u) \neq w_f(v)$ for any two distinct vertices, it follows that $w_{f_k}(u) \neq w_{f_k}(v)$. Hence f is arbitrarily distance antimagic. □

Proposition 2.2. *Let f be a distance antimagic labeling of a graph G of order n, such that $deg(u) < deg(v) \Longrightarrow w_f(u) < w_f(v)$. Then f is an arbitrarily distance antimagic labeling.*

Proof. Let u and v be two distinct vertices. Then $w_f(v) \neq w_f(u)$. Also $w_{f_k}(u) = w_f(u) + k(deg(u))$. Hence it follows that if $deg(u) = deg(v)$ then $w_{f_k}(u) \neq w_{f_k}(v)$. Now if $deg(u) < deg(v)$ then $w_f(u) < w_f(v)$. Hence $w_{f_k}(u) < w_{f_k}(v)$. Hence f is an arbitrarily distance antimagic labeling. □

Theorem 2.3. *The path P_n is arbitrarily distance antimagic for every $n \neq 3$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. We consider two cases:
Case 1 : n is odd.
Define

$$f(v_i) = \begin{cases} n & \text{if } i = 1 \\ n-1 & \text{if } i = n \\ 1 & \text{if } i = 2 \\ 2 & \text{if } i = n-1 \\ (n+1) - i & \text{if } i = 3, 4, \ldots, n-2. \end{cases}$$

Then

$$w_f(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ 2 & \text{if } i = n \\ n-2 & \text{if } i = 3 \\ n+2 & \text{if } i = n-1 \\ 2n-2 & \text{if } i = 2 \\ 2n+2-2i & \text{if } i = 4, 5, \ldots, n-2. \end{cases}$$

Case 2 : n is even.
Define

$$f(v_i) = \begin{cases} n-(i-1) & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even.} \end{cases}$$

Then

$$w_f(v_i) = \begin{cases} 2n+2-2i & \text{if } i \text{ is even} \\ 2i-2 & \text{if } i \text{ is odd, } 3 \le i \le n-1 \\ 1 & \text{if } i = 1. \end{cases}$$

In both cases the vertex weights are distinct. Hence P_n is distance antimagic. Further if $u, v \in V(P_n)$ and $deg(u) < deg(v)$, then $w_f(u) < w_f(v)$. Hence it follows from Proposition 2.2 that f is an arbitrarily distance antimagic labeling of P_n. □

Theorem 2.4. *The cycle C_n is arbitrarily distance antimagic for every $n \ge 3$, $n \ne 4$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. We consider two cases:
Case 1 : n is odd.
Define $f : V(C_n) \longrightarrow \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $1 \le i \le n$. Clearly f is a bijection and

$$w_f(v_i) = \begin{cases} n+2 & \text{if } i = 1 \\ 2i & \text{if } 2 \le i \le n-1 \\ n & \text{if } i = n. \end{cases}$$

Case 2 : n is even.
Define $f : V(C_n) \longrightarrow \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $2 \le i \le n-1$, $f(v_1) = n$ and $f(v_n) = 1$. Then

$$w_f(v_i) = \begin{cases} 3 & \text{if } i = 1 \\ n+3 & \text{if } i = 2 \\ 2i & \text{if } 3 \le i \le n-2 \\ n-1 & \text{if } i = n-1 \\ 2n-1 & \text{if } i = n. \end{cases}$$

In both the cases the vertex weights are distinct and hence C_n is distance antimagic. Now since C_n is regular it follows from Proposition 2.1 that the labeling f is an arbitrarily distance antimagic labeling. □

Handa et al. [6] have also proved the following theorem.

Theorem 2.5. *For $r \ge 2$ and $n \ne 3$, the graph rP_n admits an arbitrarily distance antimagic labeling with the highest induced vertex weight equal to $2rn - 1$.*

3. Distance antimagic labeling of join of two graphs

In this section we shall use the concept of arbitrarily distance antimagic labeling to prove that join of some classes of graphs is distance antimagic.

Theorem 3.1. *Let G_1 and G_2 be two graphs of order n_1 and n_2 with arbitrarily distance antimagic labelings f_1 and f_2 respectively and let $n_1 \le n_2$. Let $x \in V(G_1)$ be the vertex with lowest weight under f_1 and $y \in V(G_2)$ be the* *vertex with highest weight under* f_2. *If*

$$w_{f_1}(x) + \sum_{i=1}^{n_2}(n_1 + i) > w_{f_2}(y) + \Delta(G_2)n_1 + \sum_{i=1}^{n_1} i \tag{1}$$

then $G_1 + G_2$ *is distance antimagic.*

Proof. We define $f : V(G_1) \cup V(G_2) \longrightarrow \{1, 2, \ldots, n_1 + n_2\}$ by

$$f(v) = \begin{cases} f_1(v) & if\ v \in V(G_1) \\ f_2(v) + n_1 & if\ v \in V(G_2). \end{cases}$$

Clearly f is a bijection. Since f_1 and f_2 are arbitrarily distance antimagic labeling, if $u, v \in V(G_1)$ or $u, v \in V(G_2)$, then $w_f(u) \neq w_f(v)$. Now the lowest weight in G_1 under f is $w_{f_1}(x) + \sum_{i=1}^{n_2}(n_1 + i)$ and the highest weight in G_2 under f is less than or equal to $w_{f_2}(y) + \Delta(G_2)n_1 + \sum_{i=1}^{n_1} i$. Hence by hypothesis the lowest weight in G_1 is greater than the highest weight in G_2 and hence it follows that $w_f(u) \neq w_f(v)$ if $u \in V(G_1)$ and $u \in V(G_2)$. Thus f is a distance antimagic labeling of $G_1 + G_2$. $\square$

Remark 3.2. Since $n_1 \leq n_2$, inequality (1) implies the inequality

$$w_{f_1}(x) + n_1 n_2 > w_{f_2}(y) + n_1 \Delta(G_2). \tag{2}$$

Further inequality (2) implies inequality (1).

Theorem 3.3. *Let* G_1 *and* G_2 *be graphs of order at least* 4 *which are arbitrarily distance antimagic and let* $\Delta(G_1), \Delta(G_2) \leq 2$. *Then* $G_1 + G_2$ *is distance antimagic.*

Proof. Let $|V(G_1)| = n_1$, $|V(G_2)| = n_2$ and $n_1 \leq n_2$. Let f_1 and f_2 be arbitrarily distance antimagic labelings of G_1 and G_2 respectively. Now the highest vertex label in G_2 under f_2 is at most $2n_2 - 1$ and the lowest vertex label in G_1 under f_1 is at least 1. Hence inequality (2) reduces to $1 + n_1 n_2 > 2n_2 - 1 + 2n_1$. This inequality can be rewritten as $\frac{1}{2} > \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_1 n_2}$ which is obviously true since $n_1, n_2 \geq 4$. Hence the result follows from Theorem 3.1. $\square$

Corollary 3.4. *For* $n, m \geq 4$ *the graph* $P_n + P_m$ *is distance antimagic.*

Proof. Follows from Theorems 3.3 and 2.3. $\square$

Corollary 3.5. *For* $n, m \geq 5$, *the graph* $C_n + C_m$ *is distance antimagic.*

Proof. Follows from Theorems 2.4 and 3.3. $\square$

Theorem 3.6. *Let* G *be a distance antimagic graph of order* $n \geq 3$ *with distance antimagic labeling* f *such that the highest weight under* f *is less than or equal to* $\frac{n(n+1)}{2} - 3$. *Then* $G + K_3$ *is distance antimagic.*

Proof. Since K_3 is arbitrarily distance antimagic, by Theorem 3.1, it is enough to prove that inequality (1) holds. Since G is distance antimagic and $|V(G)| \geq 3$, the highest vertex weight in G is less than $\frac{n(n+1)}{2} - 3$ and the lowest vertex weight in C_3 is at least 3, inequality (1) holds. $\square$

Remark 3.7. By using the same proof technique of Theorem 3.6 it can be easily proved that if G is a distance magic graph of order n with labeling f such that the highest vertex weight under f is less than or equal to $\frac{n(n+1)}{2} - 3$, then $G + K_1$ and $G + K_2$ are distance antimagic. Hence from Theorem 2.3 and Theorem 2.4 we have the following corollary.

Corollary 3.8. *Let* $W_n = C_{n-1} + K_1$ *be the wheel on n vertices. The graph* $P_n + P_2$ *where* $n \neq 3$, $C_n + P_2$, *where* $n \neq 4$ *and* $W_m + W_n$, *where* $n, m \neq 4$ *are distance antimagic.*

4. Corona of graphs

In this section we investigate the existence of distance antimagic labeling for corona of two graphs.

Theorem 4.1. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. If there exist vertices u_i, u_j in G_2 such that $N(u_i) = N(u_j)$ then the graph $G_1 \circ G_2$ is not distance antimagic.*

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Let $H_1, H_2, \ldots, H_{n_1}$ be n_1 copies of G_2 such that v_i is adjacent to each vertex of H_i for $1 \le i \le n_1$. Let u_{ik} and u_{jk} be the vertices in H_k corresponding to the vertices u_i and u_j respectively. Then the vertices u_{ik} and u_{jk} have the same neighborhood in $G_1 \circ G_2$. Hence $G_1 \circ G_2$ is not distance antimagic. $\square$

Theorem 4.2. *Suppose G_1 is a distance magic graph of order n_1 with magic constant k. Let G_2 be a r-regular graph of order n_2 with an arbitrarily distance antimagic labeling f. Let K be the maximum weight of a vertex in G_2 under f. Suppose*

$$k + \frac{n_2}{2}(n_2 + 2n_1 + 1) > n_1 r\left(1 + \frac{(n_1 - 1)n_2}{n_1}\right) + (K + n_1). \tag{3}$$

Then $G_1 \circ G_2$ is distance antimagic.

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Let $H_1, H_2, \ldots, H_{n_1}$ be n_1 copies of G_2 such that v_i is adjacent to each vertex of H_i for $1 \le i \le n_1$. Let $V(H_i) = \{u^i_1, u^i_2, \ldots, u^i_{n_2}\}$. Let $g : V(G_1) \to \{1, 2, \ldots, n_1\}$ be a distance magic labeling of G_1. Now define a bijection $h : V(G \circ G_2) \to \{1, 2, \ldots, n_1 + n_1 n_2\}$ as follows:

$$h(v_i) = g(v_i) \text{ if } 1 \le i \le n_1 \text{ and}$$
$$h(u^i_j) = f(u_j) + n_1 + n_2(g(v_i) - 1), \text{ if } 1 \le i \le n_1,\ 1 \le j \le n_2.$$

Then

$$w_h(v_i) = k + n_2^2\left(\frac{n_1}{n_2} + (g(v_i) - 1) + \frac{n_2 + 1}{2n_2}\right), \quad \text{if} \quad 1 \le i \le n_2$$

and

$$w_h(u^i_j) = w_f(u_j) + r[n_1 + (g(v_i) - 1)n_2] + g(v_i) \text{ if } \quad 1 \le i \le n_1,\ 1 \le j \le n_2.$$

Since f is arbitrarily distance antimagic it follows that for $1 \le i \le n_1$ and $1 \le j_1, j_2 \le n_2$, $w_h(u^i_{j_1}) \ne w_h(u^i_{j_2})$. Furthermore for $i_1 \ne i_2$, $w_h(v_{i_1}) \ne w_h(v_{i_2})$.

It remains to show that $w_h(v_i) \ne w_h(u^q_j)$ for $1 \le i, q \le n_1,\ 1 \le j \le n_2$.

$$\begin{aligned}\min_{1\le i\le n_1} w_h(v_i) &= \min_{1\le i\le n_1}\left(k + n_2^2\left(\frac{n_1}{n_2} + (g(v_i) - 1) + \frac{n_2 + 1}{2n_2}\right)\right)\\ &= k + n_2^2\left(\frac{n_1}{n_2} + \frac{n_2 + 1}{2n_2}\right).\end{aligned}$$

Furthermore

$$\begin{aligned}\max_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} w_h(u^i_j) &= \max_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} \{w_f(u_j) + r[n_1 + (g(v_i) - 1)n_2] + g(v_i)\}\\ &\le K + r[n_1 + (n_1 - 1)n_2] + n_1\\ &= n_1 r\left(1 + \frac{(n_1 - 1)n_2}{n_1}\right) + (K + n_1).\end{aligned}$$

Since $k + \frac{n_2}{2}(n_2 + 2n_1 + 1) > n_1 r\left(1 + \frac{(n_1-1)n_2}{n_1}\right) + (K + n_1)$, it follows that

$$\min_{1\le i\le n_1} w_h(v_i) > \max_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} w_h(u^i_j).$$

Hence

$$w_h(v_i) \neq w_h(u_j^q)\ \ 1 \leq i, q \leq n_1,\ \ 1 \leq j \leq n_2. \qquad \square$$

Theorem 4.3. *The graph $C_4 \circ C_n$ is distance antimagic for $n \geq 9$.*

Proof. It follows from Theorem 2.4 that $C_n, n \neq 4$ admits a distance antimagic labeling and the highest weight induced by this labeling is $2n - 1$. Since C_4 is distance magic with magic constant 5, the result follows from Theorem 4.2. $\square$

5. Conclusion

In this paper the concept of arbitrarily distance antimagic labeling has been used as a tool for proving the existence of distance antimagic labelings for join and corona of two graphs. This tool can be efficiently used for proving that several classes of graphs admit distance antimagic labelings.

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