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Distance Antimagic Labeling of the Join & Corona Product of two Graphs

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Abstract

Let G be a graph of order n. Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. The weight $w_f(v)$ of a vertex with respect to f is defined by $w_f(v) = \sum_{x \in N(v)} f(x)$. The labeling f is said to be distance antimagic if $w_f(u) \neq w_f(v)$ for every pair of vertices $u, v \in V(G)$. If the graph G admits such a labeling then G is said to be a distance antimagic graph. In this paper we prove that $P_m + P_n, P_m + C_n, P_m + W_n$ and $C_m + W_n$ are distance antimagic. We have also proved that if G and H are distance antimagic graphs satisfying certain conditions, then $G + H$ is distance antimagic and if G is magic and H is arbirarily distance antimagic then $G \circ H$ is distance antimagic.

Keywords: Distance antimagic graphs, antimagic labeling.
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1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops, multiple edges or isolated vertices. For graph theoretic terminology we refer to West [5].

Most of the graph labeling methods trace their origin to the concept of β-valuation introduced by Rosa [4]. For a general overview of graph labeling we refer to the dynamic survey by Gallian [2].

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Let G be a graph of order n. Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. The weight $w_f(v)$ of a vertex v is defined by $w_f(v) = \sum_{x \in N(v)} f(x)$. The labeling f is said to be distance antimagic if $w_f(u) \neq w_f(v)$ for every pair of vertices $u, v \in V(G)$. If the graph G admits such a labeling, then G is said to be a distance antimagic graph. Many classes of graphs are known to be distance antimagic. For details one may refer to Gallian [2] and Arumugam et al. [1].

In [3] Kamatchi and Arumugam proved that the graphs P_n $(n \neq 3)$ and C_n $(n \neq 4)$ are distance antimagic. They also proved that for any graph G the corona $H = G \circ H$ is a distance antimagic graph. They posed the following problems:

1. If G is a distance antimagic graph, are $G + K_1$ and $G + K_2$ distance antimagic?

In this paper we solve the above problem in the affirmative. We also prove certain sufficient conditions for existence of distance antimagic labeling for the join and corona products of graphs

Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. Then for any positive integer k, we define a bijection $f_k : V(G) \longrightarrow \{k + 1, k + 2, \ldots, k + n\}$ by $f_k(x) = f(x) + k$. If f is distance antimagic, then weights of the vertices with respect to f_k need not be distinct.

For example for the path $P_6 = (v_1, v_2 \ldots, v_6)$, consider $f : V \longrightarrow \{1, 2, \ldots, 6\}$ defined by $f(v_i) = i$. Then

$$w_f(v_i) = \begin{cases} 2 & if \ \ i = 1 \\ 2i & if \ \ 2 \leq i \leq 5 \\ 5 & if \ \ i = 6 \end{cases}$$

Hence f is a distance antimagic labeling of P_6. However $w_{f_1}(v_6) = w_{f_1}(v_2) = 6$. But for the functions f_k with $k \geq 2$ we have $w_{f_k}(v_i) \neq w_{f_k}(v_j)$ for any two distinct vertices $v_i, v_j \in V(P_6)$.
These observations lead to the following concept of arbitrarily distance antimagic labeling, which serves as a nice tool in proving that under certain conditions the join of two graphs are distance antimagic.

Definition 1.1. *A graph G of order n is said to be arbitrarily distance antimagic if there exists a bijection $f : V \longrightarrow \{1, 2, \ldots, n\}$ such that $w_{f_k}(u) \neq w_{f_k}(v)$ for any two distinct vertices u and v and for any $k \geq 0$. The labeling f with this property is called an arbitrarily distance antimagic labeling of G.*

2 Main Results

In this section, we shall discuss the distance antimagic labelings of graphs which are arbitrarily distance antimagic.

Proposition 2.1. *Any r-regular distance antimagic graph G is arbitrarily distance antimagic.*

Proof. let f be a distance antimagic labeling of G. Since $w_{f_k}(u) = w_f(u) + rk$ and $w_f(u) \neq w_f(v)$ for any two distinct vertices, it follows that $w_{fk}(u) \neq w_{fk}(v)$. Hence f is arbitrarily distance antimagic. □

Proposition 2.2. *Let f be a distance antimagic labeling of a graph G of order n. If $w_f(u) < w_f(v)$ whenever $deg(u) < deg(v)$, then G is arbitrarily distance antimagic.*

Proof. Suppose that there exist two distinct vertices u and v such that $w_{f_k}(u) = w_{f_k}(v)$ for some $k \geq 1$. Since $w_{f_k}(u) = w_f(u) + k(\deg(u))$, it follows that

$$w_f(v) - w_f(u) = k(\deg(u) - \deg(v)). \tag{1}$$

Since $w_f(u) \neq w_f(v)$ we have $\deg(u) \neq \deg(v)$. Suppose $\deg(u) < \deg(v)$. Then $k(\deg(u) - \deg(v)) < 0$ which implies $w_f(u) > w_f(v)$, which contradicts our assumption. Hence $w_{f_k}(u) \neq w_{f_k}(v)$ for all $k \geq 1$ and G is arbitrarily distance antimagic. □

Lemma 2.3. *Let G_1 and G_2 be two graphs of order n_1 and n_2 with arbitrarily distance antimagic labelings f_1 and f_2 respectively,and let $n_1 \leq n_2$. Let $x \in V(G_1)$ be the vertex with lowest weight under f_1 and $y \in V(G_2)$ be the vertex with highest weight under f_2. If*

$$w_{f_1}(x) + \sum_{i=1}^{n_2}(n_1 + i) > w_{f_2}(y) + \Delta(G_2)n_1 + \sum_{i=1}^{n_1} i \tag{2}$$

then $G_1 + G_2$ is distance antimagic.

Proof. We define $f : V(G_1) \cup V(G_2) \longrightarrow \{1, 2, \dots, n_1 + n_2\}$ by

$$f(v) = \begin{cases} f_1(v) & if\ v \in V(G_1) \\ f_2(v) + n_1 & if\ v \in V(G_2) \end{cases}$$

Clearly f is a bijection. Since $w_{f_1}(u) \neq w_{f_1}(v)$ for any two distinct vertices $u, v \in V(G_1)$ it follows that $w_f(u) \neq w_f(v)$. Similarly since f_2 is arbitrarily distance antimagic labeling $w_f(u) \neq w_f(v)$ for any two distinct vertices $u, v \in V(G_2)$. Now the lowest weight in G_1 under f is $w_{f_1}(x) + \sum_{i=1}^{n_2}(n_1 + i)$ and the highest weight in G_2 under f is less than or equal to $w_{f_2}(y) + \Delta(G_2)n_1 + \sum_{i=1}^{n_1} i$. Hence by hypothesis the lowest weight in G_1 is greater than the highest weight in G_2 and so f is a distance antimagic labeling of $G_1 + G_2$. □

Remark 2.4. Since $n_1 \leq n_2$, inequality (2) implies the inequality

$$w_{f_1}(x) + n_1 n_2 > w_{f_2}(y) + n_1 \Delta(G_2) \tag{3}$$

Further inequality (3) implies inequality (2).

Theorem 2.5. *Let G_1 and G_2 be graphs of order at least 4 which are arbitrarily distance antimagic and let $\Delta(G_1), \Delta(G_2) \leq 2$. Then $G_1 + G_2$ is distance antimagic.*

Proof. Let $|V(G_1)| = n_1$, $|V(G_2)| = n_2$ and $n_1 \leq n_2$. Let f_1 and f_2 be arbitrarily distance antimagic labelings of G_1 and G_2 respectively. Now the highest vertex label in G_2 under f_2 is at most $2n_2 - 1$ and the lowest vertex label in G_1 under f_1 is at least 1. Hence inequality (3) reduces to $1 + n_1 n_2 > 2n_2 - 1 + 2n_1$. This inequality can be rewritten as $\frac{1}{2} > \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_1 n_2}$ which is obviously true since $n_1, n_2 \geq 4$. Hence the result follows from Lemma 2.3. □

Theorem 2.6. *The path P_n is arbitrarily distance antimagic.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. We consider two cases:
Case 1 : n is odd.

Define

$$f(v_i) = \begin{cases} n & \text{if } i = 1 \\ n-1 & \text{if } i = n \\ 1 & \text{if } i = 2 \\ 2 & \text{if } i = n-1 \\ (n+1)-i & \text{if } i = 3, 4, \ldots, n-2. \end{cases}$$

Then

$$w_f(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ 2 & \text{if } i = n \\ n-2 & \text{if } i = 3 \\ n+2 & \text{if } i = n-1 \\ 2n-2 & \text{if } i = 2 \\ 2n+2-2i & \text{if } i = 4, 5, \ldots, n-2. \end{cases}$$

Case 2 : n is even.

Define

$$f(v_i) = \begin{cases} n-(i-1) & \text{if } i \text{ is odd} \\ i-1 & if \text{ if } i \text{ is even} \end{cases}$$

Then

$$w_f(v_i) = \begin{cases} 2n+2-2i & \text{if } i \text{ is even} \\ 2i-2 & \text{if } i \text{ is odd, } 3 \le i \le n-1 \\ 1 & i = 1 \end{cases}$$

In both cases the vertex weights are distinct. Hence P_n is distance antimagic. We claim that $w_{f_k}(u) \neq w_{f_k}(v)$ for any two distinct vertices $u, v \in V(P_n)$ and for any $k \geq 1$. If $\deg(u) = \deg(v) = 2$, then $w_{f_k}(u) = w_f(u) + 2k$ and $w_{f_k}(v) = w_f(v) + 2k$. Since $w_f(u) \neq w_f(v)$, we have $w_{f_k}(u) \neq w_{f_k}(v)$. Similarily if $\deg(u) = \deg(v) = 1$, then $w_{f_k}(u) \neq w_{f_k}(v)$. If $\deg(u)$=1 and $\deg(v)$=2, then $w_{f_k}(u) =$ 1 or 2 and $w_{f_k}(v) \geq 3$. Hence $w_{f_k}(u) \neq w_{f_k}(v)$. Thus f is arbitrarily distance antimagic. □

Corollary 2.7. *For $n, m \geq 4$ the graph $P_n + P_m$ is distance antimagic.*

Proof. Follows from Theorem 2.5 and Theorem 2.6. □

Theorem 2.8. *The cycle C_n is arbitrarily distance antimagic for all $n \neq 4$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. We consider two cases:
Case 1 : n is odd.
Define $f : V(C_n) \longrightarrow \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $1 \leq i \leq n$. Clearly f is a bijection and

$$w_f(v_i) = \begin{cases} n+2 & if\ i = 1 \\ 2i & if\ 2 \leq i \leq n-1 \\ n & if\ i = n \end{cases}$$

***Case 2** : n is even .*

Define $f : V(C_n) \longrightarrow \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $2 \leq i \leq n-1$, $f(v_1) = n$ and $f(v_n) = 1$. Then

$$w_f(v_i) = \begin{cases} 3 & if\ i = 1 \\ n+3 & if\ i = 2 \\ 2i & if\ 3 \leq i \leq n-2 \\ n-1 & if\ i = n-1 \\ 2n-1 & if\ i = n \end{cases}$$

In both the cases the vertex weights are distinct and hence C_n is distance antimagic. Now since C_n is regular the result follows from Proposition 2.1.

□

Corollary 2.9. *For $n, m \geq 5$, the graph $C_n + C_m$ is distance antimagic.*

Proof. Follows from Theorem 2.8 and Theorem 2.5. □

Theorem 2.10. *Let G be a distance antimagic graph of order $n \geq 3$ with distance antimagic labeling f such that the highest weight under f is less than or equal to $\frac{n(n+1)}{2} - 3$. Then $G + K_3$ is distance antimagic.*

Proof. Since K_3 is arbitrarily distance antimagic, by Lemma 2.3, it is enough to prove that inequality (2) holds. Since G is distance antimagic and $|V(G)| \geq 3$, the highest vertex weight in G is less than $\frac{n(n+1)}{2} - 3$ and the lowest vertex weight in C_3 is at least 3, inequality (2) holds.

□

Remark 2.11. By using the same proof technique of Theorem 2.10 it can be easily proved that if G is a distance magic graph of order n with labeling f such that the highest vertex weight under f is less than or equal to $\frac{n(n+1)}{2} - 3$, then $G + K_1$ and $G + K_2$ are distance antimagic. Hence from Theorem 2.6 and Theorem 2.8 we have the following corollary.

Corollary 2.12. *$P_n + P_2$ $(n \neq 3)$, $C_n + P_2$ $(n \neq 4)$, $C_n + C_3$ $(n \neq 4)$, $C_n + W_m$ $(n, m \neq 4)$ and $W_m + W_n$ $(n, m \neq 4)$ are distance antimagic, where $W_n = C_{n-1} + K_1$ is the wheel on n vertices.*

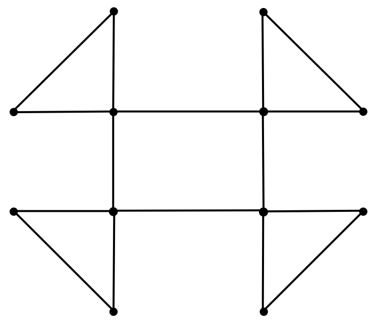

Figure 1: $C_4 \circ P_2$

3 Corona Products

Definition 3.1. *Let G and H be two graphs. The **corona product** $G \circ H$ is obtained by taking one copy of G an $|V(G)|$ copies of H and joining each vertex of the i^{th} copy of H to the i^{th} vertex of G when $1 \le i \le |V(G)|$.*

Theorem 3.2. *Let H be a graph of order m. Suppose there exists $u_i, u_j \in V(H)$ such that $N(u) = N(v)$ then the graph $G \circ H$ is not distance antimagic for any graph G.*

Proof. Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the graph H satisfying the criteria in the theorem. Let $H_1, H_2, \ldots, H_n$ be n copies of H such that for $1 \le p \le n$ wach vertex of H_p is attached to v_p. If we denote $V(H_p) = \{u_1^p, u_2^p, , u_m^p\}$, identifying u_i^p and u_j^p with u_i and u_j respectively it follows that $N(u_i^p) = N(u_j^p)$. Hence the graph $G \circ H$ is not distance antimagic. □

Theorem 3.3. *Suppose G is distance magic graph of order n with magic constant k and H is an arbitrarily distance antimagic graph of order m with an arbitrarily distance antimagic labeling f.*

Let $K = \max_{v \in V(H)} w_f(v)$ *and* $\Delta = \max_{v \in V(H)} deg(v)$.

Then $G \circ H$ is distance antimagic if the following inequality holds:

$$k + \frac{m}{2}(m + 2n + 1) \ge n\Delta\left(1 + \frac{(n-1)m}{n}\right) + (K + n) \tag{4}$$

Proof. Let $g : V(G) \to \{1, 2, \ldots, n\}$ be a distance magic labeling of G. We denote the vertex set of $G \circ H$ as follows

$$V(G \circ H) = \{v_k, u_j^i : 1 \le t, i \le n,\ 1 \le j \le m\}.$$

Such that the vertices $\{u_1^i, u_2^i, \ldots, u_m^i\}$ corresponds to the i^{th} copy of H. We define a bijective function $h : V(G \circ H) \to \{1, 2, \ldots, n + nm\}$ as follows:

$$h(u_j^i) = f(u_j) + n + mg(v_i), \qquad 1 \le i \le n,\ 1 \le j \le m$$

$$h(v_i) = g(v_i) \qquad 1 \le i \le n$$

Following are the weights induced by the above labeling:

$$\mathcal{W}_h(v_i) = k + m^2\left(\frac{n}{m} + (g(i) - 1) + \frac{m+1}{2m}\right), \qquad 1 \le i \le m$$

$$\mathcal{W}(u_j^i) = f(u_j) + deg(u_j)[n + (g(i) - 1)m] + g(v_i) \qquad 1 \le i \le n,\ 1 \le j \le m$$

Since f is arbitrarily distance antimagic it follows that for fixed i and $j_1 \neq j_2$, $\mathcal{W}(u_{j_1}^i) \neq \mathcal{W}(u_{j_2}^i)$. Also from the definition of $\mathcal{W}_h(v_i)$ it is clear that for $i_1 \neq i_2$, $\mathcal{W}_h(v_{i_1}) \neq \mathcal{W}_h(v_{i_2})$.

It remains to show that $\mathcal{W}_h(v_i) \neq \mathcal{W}_h(u_j^k)$ for $1 \le i, k \le n,\ 1 \le j \le m$.

$$\begin{aligned}\min_{1\le i\le n} \mathcal{W}_h(v_i) &= \min_{1\le i\le n}\left(k + m^2\left(\frac{n}{m} + (g(i) - 1) + \frac{m+1}{2m}\right)\right)\\ &= k + m^2\left(\frac{n}{m} + \frac{m+1}{2m}\right)\end{aligned}$$

Further

$$\begin{aligned}\max_{\substack{1\le j\le m\\1\le i\le n}} \mathcal{W}_h(u_j^i) &= \max_{\substack{1\le j\le m\\1\le i\le n}} f(u_j) + deg(u_j)[n + (g(i) - 1)m] + g(v_i)\\ &\le n\Delta\left(1 + \frac{(n-1)m}{n}\right) + (K + n)\end{aligned}$$

Therefore since

$$k + \frac{m}{2}(m + 2n + 1) \ge n\Delta\left(1 + m - \frac{\Delta - 1}{2n}\right) + (K + n)$$

It follows that

$$\min_{1\le i\le n} \mathcal{W}_h(v_i) \ge \max_{\substack{1\le j\le m\\1\le i\le n}} \mathcal{W}_h(u_j^i)$$

Therefore

$$\mathcal{W}_h(v_i) \neq \mathcal{W}_h(u_j^k)\ \ 1 \le i, k \le n,\ 1 \le j \le m.$$

This proves the result.

□

Remark 3.4. *If G is a regular distance magic graph then if f is a distance magic labeling of G then so if the labeling f_k defined by $f_k(x) = f(x) + k$. Hence combing this with the arguments in the previous result we conclude that if G is a regular distance magic graph and H is a graph satisfying the criteria of Theorem 3.3, then the graph $G \circ H$ is* ***arbitrarily*** *distance antimagic.*

Theorem 3.5. *The graph $C_4 \circ rP_n$ is arbitrarily distance antimagic for $rn \ge 10$.*

Proof. We have already defined an arbitrarily distance antimagic labeling for the graph rP_m $m \neq 3$. The highest weight induced by this labeling is $2rn-1$. Also the graph C_4 is distance magic with magic constant 5. This along with equation (4) gives us the following sufficient condition for existence of distance antimagic labeling

$$5 + \frac{rn(rn+1)}{2} + 4rn \geq 4rn + 4rn + 11$$

Solving;

$$\begin{aligned} & 5 + \frac{rn(rn+1)}{2} + 4rn \geq 4rn + 4rn + 11 \\ \Longrightarrow\ & (rn)^2 - 7rn - 7 \geq 0 \\ \Longrightarrow\ & rn \geq \left\lceil \frac{7+\sqrt{77}}{2} \right\rceil = 8 \end{aligned}$$

□

The work is still in progress to find conditions such that Corono of two arbitrarily distance antimagic graph is also distance antimagic.

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