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3 authors:

Aloysius Godinho
BITS Pilani, K K Birla Goa

Tarkeshwar Singh
BITS Pilani, K K Birla Goa

S. Arumugam
Kalasalingam University

On Nearly Distance Magic Graphs

Aloysius Godinho[*] T. Singh[†]
Department of Mathematics,
Birla Institute of Technology and Science Pilani,
K K Birla Goa Campus, NH-17B, Zuarinagar-403726,
Goa, India.

and

S. Arumugam[‡]
National Centre for Advanced Research in Discrete Mathematics,
Kalasalingam University,
Anandnagar, Krishnankoil-626126, Tamil Nadu, India.

Abstract

Let $G = (V, E)$ be a graph on n vertices. A bijection $f : V \to \{1, 2, \ldots, n\}$ is called a *nearly distance magic labeling* of G if there exist a positive integer k such that $\sum_{x \in N(v)} f(x) = k$ *or* $k+1$ for every $v \in V$. The constant k is called magic constants of the graph and the graph which admits such a labeling is called a *nearly distance magic graph*. In this paper we present several basic results on nearly distance magic graphs and compute the magic constant k in terms of the fractional total domination number of the graph.

Keywords: Distance magic graphs, nearly distance magic graphs.
2010 Mathematics Subject Classification: 05C 78.

1 Introduction

All graphs in this paper are simple graphs without isolated vertices. For graph theoretic terminology and notation we refer to Chartrand and Lesniak [3].

By a graph labeling we mean an assignment of numbers to graph elements such as vertices or edges or both. Different types of labelings have been defined by various researchers by imposing different conditions on such an assignment. For an overview on graph labeling and its recent developments we refer to Gallian [4].

Let $G = (V, E)$ be a graph of order n. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection. Define the weight of a vertex v as $w_f(v) = \sum_{x \in N(v)} f(x)$, where $N(v)$ is the open neighbourhood of v. If $w_f(v) = k$ (a constant) for every $v \in V$, then f is said to be a *distance magic labeling* of the graph G. A graph

[*]**E-mail:**p2014001@goa.bits-pilani.ac.in
[†]**E-mail:**tksingh@goa.bits-pilani.ac.in
[‡]**E-mail:**s.arumugam.klu@gmail.com

which admits a distance magic labeling is called a *distance magic graph*. The constant k is called the *distance magic constant*. The concept of distance magic labeling was originally introduced by Vilfred [8]. For an overview of known results on distance magic labeling we refer to Arumugam *et al.* [1] and Rupnow [7]. Two distance magic graphs of order 7 with magic constants 9 and 7 respectively are shown in Figure 1.

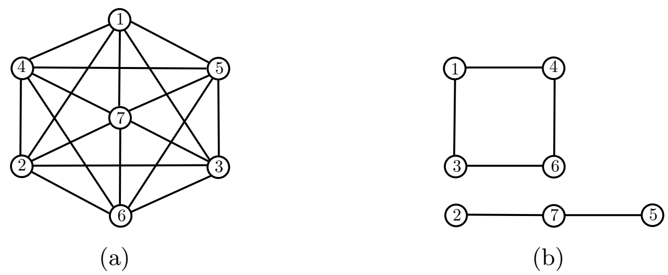

Figure 1

Definition 1.1. *Let $G = (V, E)$ be a graph without isolated vertices. A function $f : V \to [0, 1]$ is said to be a fractional total dominating function of G if for every vertex $v \in V$, $\sum_{x \in N(v)} f(x) \geq 1$. The fractional total domination number $\gamma_{ft}(G)$ is defined as $\gamma_{ft}(G) = min\ \{|f| \ :\ f\ is\ a\ fractional\ total\ dominating\ function\ of\ G\}$, where $|f| = \sum_{v \in V} f(v)$.*

At the International workshop on graph labeling (IWOGL-2010) Arumugam posed the following problem: *For a distance magic graph G of order n, is it possible to obtain two distance magic labelings f_1, f_2 with distinct magic constants k_1, k_2?* Arumugam *et al.* [2] later solved this problem by obtaining a formula for the magic constant k in terms of the fractional total domination number of the graph, thereby showing that the magic constant is independent of the labeling f. This result was also independently proved by Slater *et al.* [6].

Theorem 1.2. *If G is a distance magic graph of order n then the distance magic constant k of G is given by*

$$k = \frac{n(n+1)}{2\gamma_{ft}(G)} \tag{1}$$

Kamatchi [9] showed that the integers $4, 6, 8$ and 12 do not appear as magic constants for any distance magic graph. He posed the following problem: *Determine the set $\mathcal{S}$ of positive integers which appear as magic constants of some distance magic graph.* Froncek *et al.* [5] proved that for every $t \geq 6$ there exists a 4−regular distance magic graph with magic constant 2^t. In Theorem **??** we obtain an analogous result for nearly distance magic graphs.

2 Main Results

Consider the complete tripartite graph $G = K_{4,3,3}$. One can easily prove that $\gamma_{ft}(G) = \frac{3}{2}$. If f is a distance magic labeling of G then by (1) the magic constant $k = \frac{110}{3} \notin \mathbb{N}$. Hence the graph G is not distance magic. However, we can find a bijection $f : V \to \{1, \ldots, 10\}$ for which the weights $w_f(v) = 36$ *or* 37 (see Figure 2). It is interesting to note that the vertex weights satisfy the inequality $36 < \frac{110}{3} < 37$. This motivates the following concepts of nearly distance magic labeling.

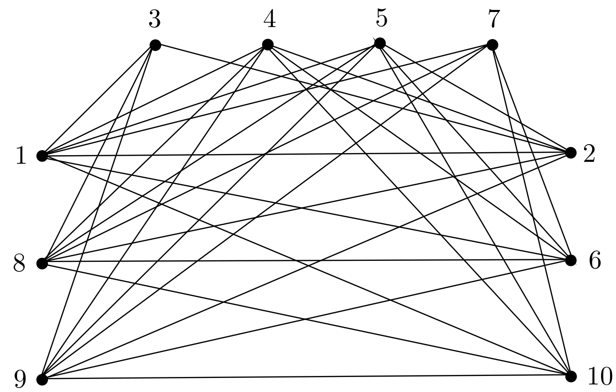

Figure 2: Nearly distance magic labeling of $K_{4,3,3}$

Definition 2.1. *Let $G = (V, E)$ be a graph on n vertices. A bijection $f : V \to \{1, 2, \ldots, n\}$ is called a nearly distance magic labeling of G if there exists a positive integer k such that $\sum_{x \in N(v)} f(x) = k$ or $k + 1$ for every $v \in V$. The constants k is called the magic constant of the graph and the graph which admits such a labeling is called a nearly distance magic graph.*

The following theorem is a generalisation of Theorem 1.2.

Theorem 2.2. *If G is a nearly distance magic with magic constant k , then*

$$k = \left\lfloor \frac{n(n+1)}{2\gamma_{ft}(G)} \right\rfloor. \tag{2}$$

Proof. Let f be a nearly distance magic labeling of G with magic constant k. Let $V = \{v_1, \ldots, v_r, \ldots v_n\}$ be the vertex set of G such that $w(v_i) = k$ for $1 \le i \le r$ and $w(v_i) = k + 1$ for $r + 1 \le i \le n$. Let $A = (a_{i,j})_{n \times n}$ be the adjacency matrix of G. Let $X = \begin{bmatrix} f(v_1) & f(v_2) & \ldots f(v_n) \end{bmatrix}^T$. Then,

$$AX = (\underbrace{k, k, \ldots, k}_{\text{r terms}}, k+1, k+1, \ldots, k+1)^T$$

Let g be a fractional total dominating function of G such that $|g| = \sum_{i=1}^{n} g(v_i) = \gamma_{ft}(G)$.

Let $Y = (g(v_1), g(v_2), \ldots g(v_n))^T$ and $AY = (l_1, l_2, \ldots l_n)^T$ with each $l_i \ge 1$

Since $X^T AY$ is a 1×1 matrix, $X^T AY = (X^T AY)^T = Y^T AX$.

$$\begin{aligned} \text{Now } Y^T AX &= Y^T (AX) \\ &= (g(v_1), g(v_2), \ldots g(v_n))(k, k, \ldots, k, k+1, k+1, \ldots, k+1)^T \\ &= k(g(v_1) + g(v_2) + \ldots + g(v_r)) + (k+1)(g(v_{r+1} + g(v_{r+2}) + \ldots + g(v_n)) = k\gamma_{ft}(G) + \sum_{i=r+1}^{n} g(v_i) \end{aligned}$$

Similarly,

$$\begin{aligned} X^T AY &= X^T (AY) \\ &= (f(v_1), f(v_2), \ldots f(v_n))(l_1, l_2, \ldots l_n)^T \\ &= \sum_{i=1}^{n} f(v_i) l_i \ge \sum_{i=1}^{n} f(v_i) = \frac{n(n+1)}{2} \end{aligned}$$

Since $X^TAY = Y^TAX$ we get $k\gamma_{ft}(G) + \sum_{i=r+1}^{n} g(v_i) \geq \frac{n(n+1)}{2}$. It follows that

$$k + \frac{\sum_{i=r+1}^{n} g(v_i)}{\gamma_{ft}(G)} \geq \frac{n(n+1)}{2\gamma_{ft}(G)} \tag{3}$$

Now define $\theta : V(G) \longrightarrow [0,1]$ by, $\theta(v) = min\ \{1, \frac{f(v)}{k}\}$. Since f is a nearly distance magic labeling with magic constant k it follows that $\sum_{x\in N(v)} \theta(x) \geq \frac{1}{k}\sum_{x\in N(v)} f(x) \geq 1$. Hence θ is a fractional total dominating function. Therefore we have,

$$\gamma_{ft}(G) \leq |\theta| = \sum_{v\in V(G)} \theta(v) \leq \frac{1}{k}\sum_{v\in V(G)} f(v) = \frac{n(n+1)}{2k}$$

Therefore

$$k \leq \frac{n(n+1)}{2\gamma_{ft}(G)} \tag{4}$$

From (3) and (4) we have $k \leq \frac{n(n+1)}{2\gamma_{ft}(G)} \leq k + \frac{\sum_{i=r+1}^{n} g(v_i)}{\gamma_{ft}(G)}$ and $0 < \frac{\sum_{i=r+1}^{n} g(v_i)}{\gamma_{ft}(G)} < 1$. Hence $k = \left\lfloor \frac{n(n+1)}{2\gamma_{ft}(G)} \right\rfloor$ □

Theorem 2.3. *The graph $G = nC_4 \cup P_2$, where $n \geq 1$ is nearly distance magic with magic constant $4n+1$.*

Proof. Let $(u_{i1}, u_{i2}, , u_{i3}, u_{i4}, u_{i1})$, $1 \leq i \leq n$ be the n copies of C_4 in G. Define $f : V \to \{1, 2, \ldots, 4n+2\}$ by

$$f(v_1) = 4n+2, \quad f(v_2) = 4n+1$$

$$\text{and } f(u_{ij}) = \begin{cases} i + n(j-1) & 1 \leq i \leq n,\ j = 1,2 \\ (7-j)n + 1 - i & 1 \leq i \leq n,\ j = 3,4 \end{cases} \tag{5}$$

Clearly f is a bijection. $w(v_2) = 4n+2$ and $w(v) = 4n+1$ for the remaining vertices. Therefore f is a nearly distance magic labeling with magic constant $4n+1$.

□

A nearly distance magic labeling of $4C_4 \cup P_2$ is given in Figure 3.

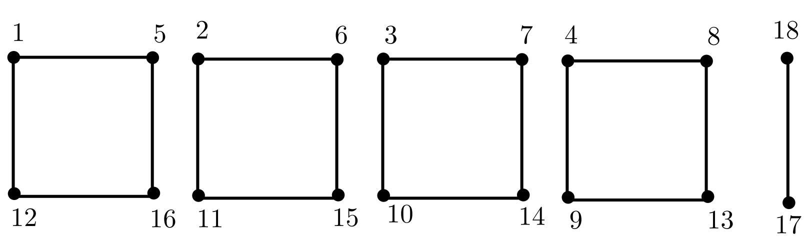

Figure 3: Nearly distance magic labeling of $4C_4 \cup P_2$

Theorem 2.4. *The graph $G = nC_4 \cup K_{2,3}$ is nearly distance magic with magic constant $4n+7$.*

Proof. Let $(u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i1})$, $1 \le i \le n$ be the n copies of C_4 in G and $\{u_{(n+1)1}, u_{(n+1)3}\}$, $\{u_{(n+1)2}, u_{(n+1)4}, u_{(n+1)5}\}$ be the bi-partition of $K_{2,3}$ .Define $g : V \to \{1, 2, \ldots, 4n+5\}$ by,

$$g(u_{ij}) = \begin{cases} i + 1 + (n+1)(j-1) & 1 \le i \le n+1,\ j = 1,2 \\ (7-j)(n+1) + 2 - i & 1 \le i \le n+1,\ j = 3,4 \\ 1 & i = n+1,\ j = 5 \end{cases} \tag{6}$$

Then $w(x) = 4(n+2)$ for $x \in \{u_{(n+1)1}, u_{(n+1)3}\}$. The weight of the remaining vertices is $4n+7$. Hence g is a nearly distance magic labeling with magic constant $4n+7$ □

A nearly distance magic labeling of $3C_4 \cup K_{2,3}$ is given in Figure 4.

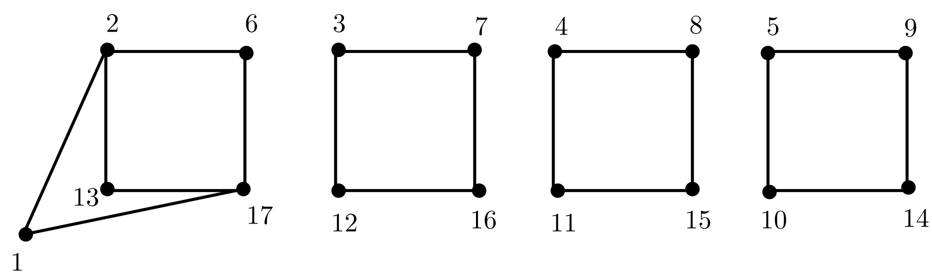

Figure 4: Nearly distance magic labeling of $3C_4 \cup K_{2,3}$

Lemma 2.5. *There is no nearly distance magic graph with magic constant 4.*

Proof. Suppose there exists a nearly distance magic graph G with magic constant 4 then $|V(G)| = 4$ *or* 5. Let $|V(G)| = 4$ and v be the vertex in G with label 4 then all vertices adjacent to v are pendent vertices and hence $w(v)$ cannot be 4 or 5 which is a contradiction. The proof is similar if $|V(G)| = 5$.

□

Observation 2.6. *It follows from Theorem 2.3 and Theorem 2.4 that for every odd integer $k \ge 5$ there exists a nearly distance magic graph with magic constant k.*

Observation 2.6, Theorems 2.3, 2.4 and Lemma 2.5 lead to the following problem:

Problem 2.1. *For any even integer $k \ne 4$, does there exist a nearly distance magic graph G with magic constant k?*

Theorem 2.7. *Let G be an r-regular graph which is not distance magic. Suppose G admits a nearly distance magic labeling f with magic constant k. Then r is odd and there are exactly $\frac{n}{2}$ vertices with weight k.*

Proof. Let f be a nearly distance magic labeling of G. Let t be the number of vertices with weight k. Hence $\sum_{v \in V} w(v) = tk + (n-t)(k+1) = n(k+1) - t$. Since G is r-regular it follows that $\sum_{v \in V} w(v) = \sum_{v \in V} \sum_{x \in N(v)} f(v) = r\frac{n(n+1)}{2}$. Hence

$$r\frac{n(n+1)}{2} = n(k+1) - t.$$

Therefore $k = \frac{r(n+1)}{2} + \frac{t}{n} - 1$. Since k is an integer it follows that r is odd, n is even and hence it follows that $t = \frac{n}{2}$.

□

Theorem 2.8. *A tree T is nearly distance magic if and only if $T \cong P_2$ or P_3 .*

Proof. Let f be a nearly distance magic labeling of T. Suppose u_1 and u_2 are pendent vertices such that $N(u_1) = \{v_1\}$, $N(u_2) = \{v_2\}$ and $v_1 \neq v_2$. Then it follows that $f(v_1) = n$ and $f(v_2) = n - 1$. Now any vertex adjacent to v_1 is a pendent vertex and hence the component which contains v_1 is a star. Therefore T is not connected which is a contradiction. Hence $T \cong K_{1,n-1}$ and moreover it follows that $T \cong P_2$ *or* P_3. □

3 Conclusion and scope

Since any distance magic graph is nearly distance magic, an interesting problem for further investigation is the construction of graphs which are nearly distance magic but not distance magic. Another possible direction for further investigation is characterisation of specific families of graphs which are nearly distance magic.

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